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## An Approach to Screening for Coulomb Wave Functions\*

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A possible method for generalizing analytic representations of Coulomb wave functions to include screening is described. One-dimensional integral representations for the Coulomb problem are replaced with two-dimensional integral representations which include the screening function in the kernel. The equations for a two-dimensional representation with exponential screening are presented. A formal solution of the equations is obtained for the nonrelativistic case.

### INTRODUCTION

Although approximate methods for computing atomic wave functions are widely used and have been very successful, it is usually the case that analytic representations of these wave functions have not been found. One possible approach to finding analytic representations, which are applicable over large ranges of atomic number, is to find a good approximate single particle potential function for which the appropriate wave equation (Schrödinger or Dirac) can be solved analytically. Gáspár [1] found a potential function which rather well approximates both the single particle potential derived from the Thomas-Fermi statistical model, and the effective single particle potentials that have resulted from a number of Hartree self-consistent field calculations. He also generalized that potential function to one which provides a rather good approximation to the potential function obtained from the statistical model with exchange and to those obtained from the self-consistent field calculations with exchange (Thomas-Fermi-Dirac and Hartree-Fock) [2]. Although the wave equations with these potentials have not been solved analytically, it may be possible to find analytic solutions for these or similar potentials. If that were done, then it might be possible to obtain a number of useful results from the solutions. For example, formulas for approximate energy eigenvalues, phase shifts, and values at the origin might be derivable. The solutions

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might be useful as good initial trial functions for self-consistent field calculations. Also, they might be helpful in the evaluation of electronic matrix elements for processes which involve atomic electrons: for example, the internal conversion process. The purpose of this note is to present some ideas for dealing with screened Coulomb potentials analytically.

The real complication of (unscreened) Coulomb wave functions, relativistic and nonrelativistic, is their involvement with confluent hypergeometric functions. However, there exist integral representations for these hypergeometric functions, whose integrands are simple elementary functions. The basic idea of the approach which is described is to generalize the Laplace integral representations of the hypergeometric functions which occur in Coulomb wave functions to include screening effects. The one-dimensional integral representations are replaced by two-dimensional representations which include the screening function in the kernel. The equations for the modulating functions are then partial differential equations, instead of ordinary differential equations, which do not involve the screening function, although the screening function does appear in the boundary conditions which the modulating functions must satisfy. Equations which determine such two-dimensional representations are developed for a class of screening functions, the simplest example of which describes exponential screening. The equations for exponential screening are presented in detail, and some attempts to solve them are discussed. A formal solution, which reduces to a one-dimensional representation, is given for the nonrelativistic case.

### THE EQUATIONS AND BOUNDARY CONDITIONS<sup>1</sup>

For a nonrelativistic particle under the influence of a central potential  $\phi(r)/r$ , with energy  $E$ , orbital angular momentum quantum number  $l$ , mass  $m$ , and charge  $e$ , the Schrödinger wave function has the form of a radial function,  $R(r)$ , times a spherical harmonic. The equation for  $R(r)$  is [3]

$$\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} + \left[ 2m \left( E - \frac{\phi}{r} \right) - \frac{l(l+1)}{r^2} \right] R = 0. \quad (1)$$

The solution of Eq. (1) which is regular at the origin can be conveniently expressed in terms of the dimensionless variable  $x = 2\sqrt{-2mEr}$  and a function  $G(x)$ , which is defined by  $R(r) = x^l e^{-1/2x} G(x)$ . The equation satisfied by  $G(x)$  is

$$M_x G \equiv x \frac{d^2 G}{dx^2} + [2(l+1) - x] \frac{dG}{dx} + \left[ \omega \frac{\phi}{C} - l - 1 \right] G = 0 \quad (2)$$

<sup>1</sup> Units are chosen throughout such that  $\hbar = c = 1$ .

where  $C = \phi(0)$  and  $\omega = -mC(-2mE)^{-1/2}$ ;  $C = -Ze^2$  for an electron in the field of an infinitely heavy nucleus of atomic number  $Z$ . The operator  $M_x$ , defined by Eq. (2), will be of interest later.

In case  $\phi/C = 1$  (unscreened Coulomb potential), the desired regular solution of Eq. (2), both for bound states ( $E < 0$ ) and continuum states ( $E > 0$ ), is a confluent hypergeometric function. In the notation of Erdélyi [4], that solution is written as

$$G(x) = \Phi(l + 1 - \omega, 2l + 2; x).$$

If

$$2l + 2 > \operatorname{Re}(l + 1 - \omega) > 0$$

which is the case if  $E > 0$ , then this function has the following integral representation:

$$G(x) = \Phi(l + 1 - \omega, 2l + 2; x) = \text{const.} \int_0^1 e^{xt} t^{l-\omega} (1-t)^{l+\omega} dt. \quad (3)$$

There is also an integral representation with the same integrand if  $E < 0$ ; in that case the integral is a contour integral.

For a relativistic particle of energy  $E$ , mass  $m$ , and charge  $e$ , under the influence of a central potential  $\phi(r)/r$ , those solutions of the Dirac equation which are simultaneously eigenfunctions of  $J^2$ ,  $J_z$ , and  $\beta(\boldsymbol{\sigma} \cdot \mathbf{L} + 1)$ —with respective eigenvalues  $j(j+1)$ ,  $m$ , and  $\kappa = (-1)^{j-l+1/2}(j + \frac{1}{2})$ —have the form

$$\begin{pmatrix} g_\kappa(r) \chi_\kappa^m \\ f_\kappa(r) \chi_{-\kappa}^m \end{pmatrix};$$

$\chi_\kappa^m$  is a two-component function of spin and angle. The equations for  $g_\kappa(r)$  and  $f_\kappa(r)$  are [5]

$$\left[ \frac{d}{dr} + \frac{1 + \kappa}{r} \right] g_\kappa(r) = \left[ E + m - \frac{\phi}{r} \right] f_\kappa(r),$$

and

$$\left[ \frac{d}{dr} + \frac{1 - \kappa}{r} \right] f_\kappa(r) = - \left[ E - m - \frac{\phi}{r} \right] g_\kappa(r). \quad (4)$$

The solution of Eqs. (4) which is regular at the origin can be conveniently expressed in terms of the dimensionless variable  $x = 2m \sqrt{1 - \epsilon^2} r$  and the functions  $G_1(x)$  and  $G_2(x)$ , which are defined by

$$g_\kappa = \frac{1}{x} \sqrt{1 + \epsilon} e^{-1/2x} x^\gamma [(C - \kappa \sqrt{1 - \epsilon^2}) G_1 + (\gamma \sqrt{1 - \epsilon^2} - C\epsilon) G_2],$$

and

$$f_\kappa = \frac{1}{x} \sqrt{1 - \epsilon} e^{-1/2x} x^\gamma [(C - \kappa \sqrt{1 - \epsilon^2}) G_1 - (\gamma \sqrt{1 - \epsilon^2} - C\epsilon) G_2],$$

where  $\epsilon = E/m$  and  $\gamma = \sqrt{\kappa^2 - C^2}$ . The equations satisfied by  $G_1(x)$  and  $G_2(x)$  are

$$\begin{aligned}
 x \frac{dG_1}{dx} &= \left[ x + \frac{1}{2}(\omega_1 - \omega_2 - 1) \frac{\phi}{C} - \frac{1}{2}(\omega_1 + \omega_2 + 1) \right] G_1 \\
 &\quad + \frac{\omega_2 + 1}{1 - \eta} \left[ \frac{\phi}{C} - \eta \right] G_2, \\
 \text{and} \\
 x \frac{dG_2}{dx} &= - \left[ \frac{1}{2}(\omega_1 - \omega_2 - 1) \frac{\phi}{C} + \frac{1}{2}(\omega_1 + \omega_2 + 1) \right] G_2 \\
 &\quad + \frac{\omega_1}{1 + \eta} \left[ \frac{\phi}{C} + \eta \right] G_1, \quad (5)
 \end{aligned}$$

where

$$\omega_1 = \gamma + C \frac{\epsilon}{\sqrt{1 - \epsilon^2}}, \quad \omega_2 = \gamma - C \frac{\epsilon}{\sqrt{1 - \epsilon^2}} - 1, \quad \eta = \frac{2\kappa\epsilon}{\omega_1 - \omega_2 - 1},$$

and

$$1 - \frac{1}{\eta^2} \frac{\omega_1(\omega_2 + 1)}{\kappa^2}.$$

In analogy with the nonrelativistic case, if  $\phi/C = 1$ , then the functions  $G_1$  and  $G_2$  which are regular at the origin are confluent hypergeometric functions. For continuum solutions ( $\epsilon > 1$ ), their integral representations are:

$$G_1(x) = \Phi(\omega_1 + 1, \omega_1 + \omega_2 + 2; x) = \xi \int_0^1 e^{xt} t^{\omega_1} (1 - t)^{\omega_2} dt,$$

and

$$\begin{aligned}
 G_2(x) &= \Phi(\omega_1, \omega_1 + \omega_2 + 2; x) = \xi \frac{\omega_1}{\omega_2 + 1} \int_0^1 e^{xt} t^{\omega_1 - 1} (1 - t)^{\omega_2 + 1} dt. \\
 (\xi &= \text{const.}) \quad (6)
 \end{aligned}$$

In the following, we restrict ourselves to continuum solutions ( $E > 0$  or  $\epsilon > 1$ ), and look for generalizations of the integral representations given by Eqs. (3) and (6) to include screening.

The technique which is used for constructing integral representations with the Laplace kernel,  $e^{xt}$ , requires that the equation to be solved can be brought into a form in which the coefficients are polynomials in the independent variable.<sup>2</sup> However, we want to treat screening functions  $\phi$ , for example an exponential, for which it is not possible to bring Eqs. (2) and (5) into that form. In order to overcome this difficulty, we consider more general kernels which are functions of an additional variable. We first consider the

<sup>2</sup> For a discussion of one-dimensional integral representations, see [6].

nonrelativistic case, and look for a representation of the function  $G(x)$  in Eq (2) of the form

$$G(x) = \int_R K(x, t, u) v(t, u) dt du. \quad (7)$$

The kernel  $K(x, t, u)$  is a specific function of the three variables which is chosen a priori. The idea is to choose a kernel for which the modulating function,  $v(t, u)$ , and the region of integration,  $R$ , are relatively simple. In order to obtain a differential equation for the modulating function, we require that  $K(x, t, u)$  satisfy the relation

$$M_x K(x, t, u) = M_{tu} K(x, t, u), \quad (8)$$

where  $M_{tu}$  is an operator which is a function of the variables  $t$  and  $u$ , and of the derivatives with respect to  $t$  and  $u$ , but which is independent of  $x$  and its derivatives;  $M_x$  is defined by Eq. (2). We require that  $M_{tu}$  have an adjoint,  $\tilde{M}$ ; then there exists a vector  $\mathbf{B}$  such that

$$\begin{aligned} v(t, u) M_x K(x, t, u) &= v(t, u) M_{tu} K(x, t, u) \\ &= K(x, t, u) \tilde{M} v(t, u) + \nabla \cdot \tilde{\mathbf{B}}. \end{aligned} \quad (9)$$

The components of  $\mathbf{B}$ ,  $B_t$  and  $B_u$ , along the positive  $t$  and  $u$  axes respectively, are expressed in terms of  $K(x, t, u)$  and  $v(t, u)$ . If the boundary of the region  $R$  is independent of  $x$ , then, using Eq. (9), we can rewrite Eq. (2) as

$$M_x G(x) = \int_R K(x, t, u) [\tilde{M} v(t, u)] dt du + \int_C \mathbf{B} \cdot \mathbf{n} ds = 0, \quad (10)$$

where  $C$  is the boundary of  $R$ ,  $\mathbf{n}$  is the outward normal to  $C$ , and  $ds$  is the differential element of arc length along  $C$ .

In the following, we shall allow the boundary  $C$  to be a function of  $x$ , specified by  $u = \psi(x, t)$ . We assume that the function  $\psi(x, t)$  is a double-valued function of  $t$ , and that the extreme values of  $t$  which bound  $C$  are independent of  $x$ . In this case, the equation for  $G(x)$ , Eq. (2), can still be written in the form given by Eq. (10), except that  $B_t$  and  $B_u$ , the components of  $\mathbf{B}$ , now depend on  $\psi(x, t)$ .

We choose the modulating function,  $v(t, u)$ , and the boundary  $C$  by requiring

$$\tilde{M} v(t, u) = 0, \quad (11a)$$

and

$$\int_C \mathbf{B} \cdot \mathbf{n} ds = 0. \quad (11b)$$

For convenience we replace Eq. (11b) with the more restrictive condition

$$\mathbf{B} \cdot \mathbf{n} = 0 \quad \text{along} \quad C. \quad (11c)$$

With these conditions,  $G(x)$  will obviously satisfy Eqs. (2) and (10).

The same procedure is applicable in the relativistic case, in which we look for a solution of Eqs. (5). We replace the operators  $M_x$ ,  $M_{tu}$ , and  $\tilde{M}$  with three corresponding  $2 \times 2$  matrices, and we replace the functions  $G(x)$ ,  $v(t, u)$ , and  $\mathbf{B}$  with corresponding two-component column vectors:

$$G(x) = \begin{pmatrix} G_1(x) \\ G_2(x) \end{pmatrix}, \quad v(t, u) = \begin{pmatrix} v_1(t, u) \\ v_2(t, u) \end{pmatrix}, \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} \mathbf{B}^{(1)} \\ \mathbf{B}^{(2)} \end{pmatrix}.$$

Equations (7)-(11) remain valid when this is done.

One way of satisfying  $\mathbf{B} \cdot \mathbf{n} = 0$ , Eq. (11c), is to choose a curve such that  $\mathbf{B} \perp \mathbf{n}$ . If  $u = \psi(x, t)$  represents a curve such that  $\mathbf{B} \perp \mathbf{n}$ , then  $\psi(x, t)$  must satisfy

$$\frac{\partial \psi}{\partial t} = \frac{B_u}{B_t} \quad (\text{nonrelativistic case}), \quad (12a)$$

or

$$\frac{\partial \psi}{\partial t} = \frac{B_u^{(1)}}{B_t^{(1)}} = \frac{B_u^{(2)}}{B_t^{(2)}} \quad (\text{relativistic case}). \quad (12b)$$

We must choose a kernel,  $K(x, t, u)$ , with which Eq. (8) can be satisfied. The one which we choose, a generalization of the Laplace kernel, is

$$K(x, t, u) = e^{\pi i} e^{(\phi(x)/C)u}. \quad (13)$$

Equation (8) can be satisfied with this kernel if we restrict  $\phi(x)$  to obey the equation

$$\frac{d\phi}{dx} = \sum_{n=0}^N \sum_{m=0}^M a_{nm} x^n \phi^m,$$

where  $a_{nm}$  are constants. This is guaranteed by the relations

$$x^n K = \frac{\partial^n K}{\partial t^n}, \quad \text{for} \quad n \geq 0, \quad (14a)$$

$$\phi^m K = \frac{\partial^m K}{\partial u^m}, \quad \text{for} \quad m \geq 0, \quad (14b)$$

and

$$\frac{\partial K}{\partial x} = \left[ t + \frac{u}{C} \sum_{n=0}^N \sum_{m=0}^M a_{nm} \frac{\partial^n}{\partial t^n} \frac{\partial^m}{\partial u^m} \right] K. \quad (14c)$$

The simplest  $\phi(x)$  of this form, which would be of interest as a screening function, is the exponential

$$\phi(x) = e^{-\lambda x}. \quad (15)$$

We now limit our considerations to this exponential screening.

The equations  $\tilde{M}v = 0$  and the expressions for  $B_t$  and  $B_u$  are presented in Eqs. (16) and (17) for exponential screening.

*Nonrelativistic Case*

$$\left[ t + \lambda \frac{\partial}{\partial u} u \right]^2 \frac{\partial v}{\partial t} - \left[ t + \lambda \frac{\partial}{\partial u} u \right] \left[ 2l + \frac{\partial}{\partial t} \right] v + \left[ \omega \frac{\partial}{\partial u} + l \right] v = 0, \quad (16a)$$

$$B_t = K \left\{ -t(1-t)v - \lambda \frac{\partial}{\partial u} [(1-2t)uv] + \lambda^2 \left[ \frac{\partial^2}{\partial u^2} (u^2v) - \frac{\partial}{\partial u} (uv) \right] \right\},$$

$$\begin{aligned} B_u = K \left\{ \omega v + \lambda [x(1-2t) - 2(l+1)] \psi v \right. \\ + \lambda^2 x \left[ \left( 1 + \frac{\phi}{C} \psi \right) \psi v - \frac{\partial}{\partial u} (u^2v) \right] \\ + x \left[ \left( -\frac{\phi}{C} v + \frac{\partial v}{\partial u} \right) \left( \frac{\partial \psi}{\partial x} \right)^2 + 2 \left( t - \lambda \psi \frac{\phi}{C} \right) v \frac{\partial \psi}{\partial x} + v \frac{\partial^2 \psi}{\partial x^2} \right] \\ \left. + [2(l+1) - x] v \frac{\partial \psi}{\partial x} \right\}. \end{aligned} \quad (16b)$$

*Relativistic Case*

$$\begin{aligned} -\lambda \frac{\partial}{\partial t} \frac{\partial}{\partial u} uv_1 + \frac{\partial}{\partial t} (1-t)v_1 + \frac{1}{2}(\omega_1 - \omega_2 - 1) \frac{\partial v_1}{\partial u} + \frac{\omega_2 + 1}{1 - \eta} \frac{\partial v_2}{\partial u} \\ + \frac{1}{2}(\omega_1 + \omega_2 + 1)v_1 + (\omega_2 + 1) \frac{\eta}{1 - \eta} v_2 = 0, \\ -\lambda \frac{\partial}{\partial t} \frac{\partial}{\partial u} uv_2 - \frac{\partial}{\partial t} tv_2 - \frac{1}{2}(\omega_1 - \omega_2 - 1) \frac{\partial v_2}{\partial u} + \frac{\omega_1}{1 + \eta} \frac{\partial v_1}{\partial u} \\ + \frac{1}{2}(\omega_1 + \omega_2 + 1)v_2 - (\omega_1) \frac{\eta}{1 + \eta} v_1 = 0. \end{aligned} \quad (17a)$$

$$\begin{aligned} B_t = K \begin{pmatrix} (t-1)v_1 + \lambda \frac{\partial}{\partial u} uv_1 \\ tv_2 + \lambda \frac{\partial}{\partial u} uv_2 \end{pmatrix}, \\ B_u = -K \begin{pmatrix} \frac{1}{2}(\omega_1 - \omega_2 - 1)v_1 + \frac{\omega_2 + 1}{1 - \eta} v_2 + \lambda x \psi v_1 - x \frac{\partial \psi}{\partial x} v_1 \\ -\frac{1}{2}(\omega_1 - \omega_2 - 1)v_2 + \frac{\omega_1}{1 + \eta} v_1 + \lambda x \psi v_2 - x \frac{\partial \psi}{\partial x} v_2 \end{pmatrix}. \end{aligned} \quad (17b)$$

In the expressions for  $B_t$  and  $B_u$ , derivatives with respect to  $u$  are to be evaluated at  $u = \psi(x, t)$ . Also, to  $B_t$  and  $B_u$  may be added any functions  $B_t'$  and  $B_u'$  respectively, such that

$$\frac{\partial B_t'}{\partial t} + \frac{\partial B_u'}{\partial u} = 0.$$

Equations (2) and (5) will be solved if we can solve the problem of finding a modulating function,  $v(t, u)$ , and a boundary curve,  $C$ , such that  $\tilde{M}(v) = 0$  (Eqs. (16a) and (17a)) and  $\mathbf{B} \cdot \mathbf{n} = 0$  along  $C$ .

### NONRELATIVISTIC CASE

The general solution of  $\tilde{M}v(t, u) = 0$ , Eq. (16a), must certainly be difficult to obtain in closed form. However, the general solution can be expressed as a power series in  $\lambda$ . An appropriate way of doing this is to allow the coefficients<sup>3</sup> to be functions of independent variables  $t + \lambda$  and  $u$ :

$$v(t, u) = \sum_{n=0}^{\infty} \lambda^n v_n(t + \lambda, u). \quad (18)$$

Whereas the equation for  $v(t, u)$  is a third order partial differential equation, the equation for  $v_n(t + \lambda, u)$ , in terms of  $v_{n-1}(t + \lambda, u)$  and  $v_{n-2}(t + \lambda, u)$ , is an inhomogeneous first order equation:

$$(t + \lambda)(1 - t - \lambda) \frac{\partial v_n}{\partial t} - \omega \frac{\partial v_n}{\partial u} = l(1 - 2t - 2\lambda) v_n + F_n(t + \lambda, u),$$

where<sup>3</sup>

$$\begin{aligned} F_n(t + \lambda, u) = & -\lambda \left\{ (1 - 2t - 2\lambda) u \frac{\partial}{\partial u} \frac{\partial v_{n-1}}{\partial t} + 2lu \frac{\partial v_{n-1}}{\partial u} \right. \\ & \left. + \lambda^2 \left( u^2 \frac{\partial^2}{\partial u^2} \frac{\partial v_{n-2}}{\partial t} + u \frac{\partial}{\partial u} \frac{\partial v_{n-2}}{\partial t} \right) \right\}. \end{aligned} \quad (19b)$$

This equation can be solved by means of the theory of characteristics [7]; the solution is

$$\begin{aligned} v_n(t + \lambda, u) = & [(t + \lambda)(1 - t - \lambda)]^l \left\{ \int^{t+\lambda} [t'(1 - t')]^{-l-1} \right. \\ & F_n \left[ t', u - \ln \left( \frac{1 - t - \lambda}{t + \lambda} \right)^\omega + \ln \left( \frac{1 - t'}{t'} \right)^\omega \right] dt' \\ & \left. + g \left[ u - \ln \left( \frac{1 - t - \lambda}{t + \lambda} \right)^\omega \right] \right\}, \end{aligned} \quad (20)$$

<sup>3</sup> The only dependence of  $v_n(t + \lambda, u)$  on  $\lambda$  is in the combination  $t + \lambda$ .  $v(t, u)$  and  $F_n(t + \lambda, u)$  have an additional dependence on  $\lambda$  not indicated by the notation.



where  $g$  is an arbitrary function. We see that the variable  $u$  only appears in the combination

$$u = \ln \left( \frac{1 - t - \lambda}{t + \lambda} \right)^\omega.$$

If we allow  $v_0(t + \lambda, u)$  to be a function of  $t + \lambda$  only, then all of the other coefficients vanish, and we obtain one particular solution of Eq. (16a) in closed form:

$$v(t, u) = v_0(t + \lambda, u) = [(t + \lambda)(1 - t - \lambda)]^l. \quad (21)$$

This solution is a very special one, in that it is not a function of  $u$ . It is possible that less special solutions in closed form can be found, solutions which may be more suitable to the problem of solving the Schrödinger equation. However, with this  $u$  independent solution, Eq. (21), we can define a curve  $C$  along which  $\mathbf{B} \cdot \mathbf{n} = 0$ .

If  $v(t, u)$  is  $u$  independent, then  $B_t$  and  $B_u$ , Eqs. (16b), are considerably simplified:

$$B_t = -(t + \lambda)(1 - t - \lambda)vK,$$

and

$$\begin{aligned} B_u = & \left\{ \omega + \lambda[x(1 - 2(t + \lambda)) - 2(l + 1)]\psi + \lambda^2 x(1 + e^{-\lambda x})\psi \right. \\ & \left. + x \left[ e^{-\lambda x} \left( \frac{\partial \psi}{\partial x} \right)^2 + 2\left(t - \frac{1}{2} - \lambda e^{-\lambda x}\psi\right) \frac{\partial \psi}{\partial x} + \frac{\partial^2 \psi}{\partial x^2} \right] + 2(l + 1) \frac{\partial \psi}{\partial x} \right\} vK. \end{aligned} \quad (22)$$

Likewise, the equation which determines the curves along which  $\mathbf{B} \perp \mathbf{n}$ , Eq. (12a), is also simplified. In terms of a function  $f(x, t)$ , defined by

$$f(x, t) = e^{-\lambda x} \psi(x, t),$$

that equation is

$$\begin{aligned} -(t + \lambda)(1 - t - \lambda) \frac{\partial f}{\partial t} = & \omega e^{-\lambda x} + 2(l + 1) \frac{\partial f}{\partial x} \\ & + x \left\{ \frac{\partial^2 f}{\partial x^2} + \left( \frac{\partial f}{\partial x} \right)^2 - [1 - 2(t + \lambda)] \frac{\partial f}{\partial x} \right\}. \end{aligned} \quad (23)$$

In this equation, and in the remainder of this section, we will consider  $(t + \lambda)$  to be a real variable.<sup>4</sup>

First consider the case  $\lambda = 0$  (Coulomb field). A very simple solution of Eq. (23) is

$$f(x, t) = \psi(x, t) = \omega \ln \left( \frac{1 - t}{t} \right), \quad (24)$$

<sup>4</sup> For cases of physical interest,  $e^{-\lambda x}$  is real. Therefore,  $\lambda$  is imaginary when  $E > 0$ .

which is independent of  $x$ . Thus,  $\mathbf{B}$  and  $\mathbf{n}$  are perpendicular along the curve  $u = \omega \ln((1-t)/t)$ , which is asymptotic to the lines  $t = 0$  and  $t = 1$  in the plane determined by the  $t$  axis and the imaginary  $u$  axis.<sup>5</sup> Furthermore, when  $t = 0$  or  $t = 1$ , then  $B_t = 0$ ; and, when  $\text{Re } u = -\infty$ , then  $B_t = B_u = 0$ . Therefore, the condition  $\mathbf{B} \cdot \mathbf{n} = 0$  is satisfied along a curve,  $C$ , which bounds a region,  $R$ , consisting of two intersecting planes in the space of the real variable  $t$  and complex variable  $u$ . One part of  $C$  lies in the plane of the  $t$  axis and the imaginary  $u$  axis, and it consists of the curve  $u = \omega \ln[(1-t)/t]$  and the line  $t = 0$ . The other part of  $C$  lies in the plane perpendicular to the imaginary  $u$  axis which is defined by  $u = -i\infty$ ; it consists of the two semi-infinite lines  $t = 0$  and  $t = 1$  which run from  $\text{Re } u = 0$  to  $\text{Re } u = -\infty$ , plus a line at  $\text{Re } u = -\infty$  which joins the lines  $t = 0$  and  $t = 1$ : The three lines form an infinitely long  $U$ -shaped curve. With this region  $R$ , the function  $G(x)$ , Eq. (7), for  $\lambda = 0$ , is

$$\begin{aligned} G(x) &= \int_0^1 e^{xt} [t(1-t)]^l \int_{-\infty}^{\omega \ln(1-t)/t} e^u du dt \\ &= \int_0^1 e^{xt} t^{l-\omega} (1-t)^{l+\omega} dt, \end{aligned} \quad (25)$$

which agrees with Eq. (3).

For arbitrary  $\lambda$ , we can similarly define a curve  $C$  and a region  $R$  such that  $\mathbf{B} \cdot \mathbf{n} = 0$ . We must require that  $f(x, t)$  reduce to the function given by Eq. (24) when  $\lambda = 0$ , and that

$$[(t + \lambda)(1 - t - \lambda)]^{l+1} e^{f(x, t)} = 0, \quad \text{for } t + \lambda = 0 \quad \text{and} \quad t + \lambda = 1. \quad (26)$$

The definitions of  $C$  and  $R$  are the same as those given for  $\lambda = 0$ , except that the lines  $t = 0$  and  $t = 1$  are replaced with the lines  $t + \lambda = 0$  and  $t + \lambda = 1$ , and the curve  $u = \omega \ln[(1-t)/t]$  is replaced with  $u = \psi(x, t) = e^{\lambda x} f(x, t)$ . Then the function  $G(x)$  is given by

$$\begin{aligned} G(x) &= e^{-\lambda x} \int_0^1 e^{xt} [t'(1-t')]^l \int_{-\infty}^{e^{\lambda x} f(x, t'-\lambda)} u e^{-\lambda x} du dt' \\ &= \int_0^1 e^{xt} [t'(1-t')]^l e^{f(x, t'-\lambda)} dt'. \end{aligned} \quad (27)$$

The boundary condition which must be placed on the solution of Eq. (23) in determining  $f(x, t)$  is that the asymptotic form of Eq. (27) in the variable  $x$  be the asymptotic form of  $G(x)$  which is appropriate to the physical problem. The function  $f(x, t)$  which leads to the desired  $G(x)$  is not unique.

<sup>5</sup> Since  $E > 0$ ,  $\omega$  is imaginary.

In this manner, we are able to satisfy the conditions which were set for a two-dimensional representation. However, the solution which we have is a formal one, in the sense that we still have to solve the equation for  $f(x, t)$ . Indeed, if  $G(x)$  of the form given by Eq. (27) be substituted into Eq. (2) directly, then it can easily be seen that Eqs. (23) and (26) must be satisfied. Nevertheless, there is the possibility that either  $f(x, t)$  can be found in a more tractable form than a power series<sup>6</sup> in  $x$ , or that a different modulating function can be found, for which the boundary of the region of integration can be simply expressed.

### RELATIVISTIC CASE

Attempts to solve Eqs. (17a) and to find a corresponding region of integration which would lead to a solution of the Dirac equation have been unsuccessful. Power series for  $v(t, u)$  can be generated from Eqs. (17a), and there is also a closed form solution which is independent of  $u$ . The coefficients of a power series in  $u$  can be expressed in terms of  $v(t, 0)$  and its derivatives with respect to  $t$ , without introducing integrals. However, an appropriate region of integration has not been found for any of these functions.

The functions  $v_1$  and  $v_2$  which are independent of  $u$  are degenerate hypergeometric functions [5] which can be written as

$$\begin{aligned}
 v_1 = & A(t + \lambda)^{1/2(\omega_1 + \omega_2 + 1)} \sum_{n=0}^{|\kappa| - 1} a_n(t + \lambda)^n \\
 & - B(\omega_2 + 1) \frac{\eta}{1 - \eta} (1 - t - \lambda)^{1/2(\omega_1 + \omega_2 + 1)} \sum_{n=0}^{|\kappa|} b_n(1 - t - \lambda)^{n-1} \\
 \text{and} \\
 v_2 = & A\omega_1 \frac{\eta}{1 + \eta} (t + \lambda)^{1/2(\omega_1 + \omega_2 + 1)} \sum_{n=0}^{|\kappa|} b_n(t + \lambda)^{n-1} \\
 & + B(1 - t - \lambda)^{1/2(\omega_1 + \omega_2 + 1)} \sum_{n=0}^{|\kappa| - 1} a_n(1 - t - \lambda)^n, \quad (28a)
 \end{aligned}$$

<sup>6</sup>  $f(x, t)$  can be obtained from Eq. (23) as a power series in  $x$ ,  $f(x, t) = \sum_{n=0}^{\infty} f_n(t) x^n$ , where  $f_{n+1}$  is expressed in terms of  $f_0, \dots, f_n$ , and  $df_n/dt$ .

where  $A$  and  $B$  are constants;  $a_n$  and  $b_n$  are defined by

$$a_0 = 1, \quad (n^2 - \kappa^2) b_n = -\frac{1}{2}(\omega_1 + \omega_2 + 1 + 2n) a_n,$$

and

$$(n+1) b_{n+1} = \frac{(n^2 - \kappa^2)}{\frac{1}{2}(\omega_1 + \omega_2 + 1 + 2n)} \quad b_n = -a_n. \quad (28b)$$

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